Universität Konstanz
Fachbereich Mathematik und Statistik
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Exercises for the course "Linear Algebra I"

Sheet 5

Hand in your solutions on Thursday, 28. November 2019, 09:55, in the Postbox of your Tutor in F4. Please try to write your solutions clear and readable, write your name and the name of your tutor on each sheet, and staple the sheets together.

Exercise 5.1 *Beweismechanikaufgabe* *(4 points)*

Stick to the rules of Beweismechanik. Hand in your solutions separately in the letterbox with the label "Beweismechanikaufgaben" (letterbox 18). Mathematical notions and symbols which you may not know yet can also be found in the PDF-file Beweismechanik in our Ilias folder.

Let X, Y be sets and $f : X \to Y$ a map from X to Y. Show that $f[A \cup B] = f[A] \cup f[B]$ holds for all subsets $A, B \subset X$.

Exercise 5.2 *(6 points)*

Let K be a field and $n \in \mathbb{N}$. In what follows we refer, as usual, to the componentwise addition and the row-by-column multiplication of matrices.

(a) We consider the set

$$M_1 := \{M \in M_{n\times n}(K) \colon A \text{ ist invertible}\}.$$

Show that $(M_1, \cdot)$ is a group. For what reasons is $(M_1, +)$ not a group?

(b) Prove or disprove that

$$M_2 := \left\{ M \in M_{n\times n}(K) \mid \exists\, a \in K : M = \begin{pmatrix} a & a & \cdots & a & a \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right\}$$

is a field for any choice of K and n.

(c) Is

$$M_3 := \left\{ M \in M_{n\times n}(K) \mid \exists\, a \in K \colon M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \vee M = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \right\}$$

a field for any choice of K?

Exercise 5.3 *(4 points)*

Let K be a field, let $l, m, n \in \mathbb{N}$ and let $A \in M_{l\times m}(K)$ and $B \in M_{m\times n}(K)$. In what follows, if M is a matrix, we denote by C_M^j the j-th column of M, and by R_M^i the i-th row of M.

(a) Show that the j-th column of AB equals the product of A and the j-th column of B, i.e. show that

$$C_{AB}^j = A \cdot C_B^j.$$

(b) Show that the columns of AB are a linear combination of the columns of A.

(c) Show that the i-th row of AB equals the product of the i-th row of A and B, i.e. show that

$$R_{AB}^i = R_A^i \cdot B.$$

(d) Show that the rows of AB are a linear combination of the rows of B.

Exercise 5.4 *(2 points)*

In this excercise you conclude the proof of Satz 8.3.

Let e be an elementary row transformation of type 2, let $E = e(I_m)$ and let A be an $m \times n$-matrix over a field K. Show that $e(A) = EA$.